Performance and Stability Analysis of LP-MPC and QP-MPC Cascade Control Systems

Chao-Ming Ying and Babu Joseph
Dept. of Chemical Engineering, Washington University, St. Louis, MO 63130

Model predictive control (MPC) is used extensively in industry to optimally control constrained, multivariable processes. For nonsquare systems (with more inputs than outputs), extra degrees of freedom can be used to dynamically drive the process to its economic optimum operating conditions. This is accomplished by cascading a local linear programming (LP) or quadratic programming (QP) controller using steady-state models. Such a cascade control scheme (LP-MPC or QP-MPC) continuously computes and updates the set points used by the lower-level MPC algorithm. While this methodology has been in use by industry for many years, its properties have not been addressed in the literature. The properties of such cascaded MPC systems are analyzed from the point of view of implementation strategies, stability properties, and economic and dynamic performance. Some theoretical results on stability are derived along with a case study involving the Shell control problem.

Introduction

Model predictive control (MPC) is one of the main control strategies employed in applications of advanced control in the process industries (Martin et al., 1986; Lee and Cooley, 1997). Cutler and Ramaker (1979) used the term dynamic matrix control (DMC) and Richalet et al. (1978) used the term model predictive heuristic control (MPHC) to describe this approach. Garcia and Morari (1985a,b) realized the linear nature of the DMC algorithm and coined the term IMC (internal model control) to describe some of its variations and reported some theorems on the stability of similar algorithms. Later, this was extended to quadratic dynamic matrix control (QDMC) (Garcia and Morshedi, 1984), which considered constraints on input and output variables.

The initial articles on MPC include the output error $\|y - y_{set}\|^2$ and input move suppression $\|\Delta u\|^2$ terms in the objective function (Cutler and Ramaker, 1979). There are no input error terms $\|u - u_{set}\|^2$ in the objective function. Historically, linear optimal control strategies have incorporated a cost penalty for the input variables in the objective function (Kwakernaak and Sivan, 1972; Kailath, 1980). Inclusion of such a term in MPC formulation leads to steady-state offsets. Cutler and Ramaker (1979) recognized this and chose to pose the optimization problem in terms of changes in the control moves rather than the actual control variable, thus automatically incorporating integral action in the controller and eliminating steady-state offsets. Garcia and Morari (1985b) did include a cost for u in their DMC objective function along the lines of classic linear-quadratic-Gaussian (LQG) formulations of optimal control theory.

In many applications, when the number of inputs exceeds the number of controlled variables, it is desirable (from an economic point of view) to also try to achieve some set points on the manipulated input variables. Typically, a real-time optimizer is used to compute the economic target values for both output and some input variables usually in a period of a few hours to a few days (Marlin and Hrymak, 1996; Miletic and Marlin, 1996; Forbes and Marlin, 1996). One way to achieve these set points would be including a cost term $\|u - u_{set}\|^2$ in the objective function used by the MPC algorithm. However, this has two drawbacks: First, the presence of active constraints can lead to loss of degrees of freedom which in turn can result in steady-state offsets in the systems. This is because the MPC algorithm will distribute the final steady-state errors between the target values for both u and y. Second, the presence of external disturbances can cause the economic optimum to shift, say, for example, from one

Correspondence concerning this article should be addressed to B. Joseph.

set of constraints to another, thus changing the target values for input variables. To address this issue, some authors (Morshedi et al., 1985; Brosilow and Zhao, 1988; Yousfi and Tournier, 1991; Harkins, 1991; Muske and Rawlings, 1993) suggested a modification of the original MPC algorithm which consists of a steady-state controller in a cascade with the classical MPC controller. The outer controller continuously updates the set points used by the MPC controller (this modification is referred to as *two-stage MPC* to distinguish it from the classical MPC algorithm). The outer loop can be a Linear Program (LP-MPC) or a Quadratic Program (QP-MPC). Forbes and Marlin (1994) discussed the model accuracy requirement for the steady-state optimization layers. Vuthandam et al. (1995) proposed an alternative approach called EQDMC (QDMC with end condition). Their approach of removing offset is to make the input satisfy an end condition at the end of the control horizon.

The LP-MPC structure is widely used in industry since most practical problems involve nonsquare MIMO systems. The objective of this article is to analyze some of the properties of the two-stage MPC and to highlight its advantages using the Shell control problem. Since the dynamic performance characteristics of two-stage MPC have not been addressed in the literature, in this article it is analyzed and compared with single-stage MPC. A commercial version of the scheme is used by the Dynamic Matrix Control Corporation (Harkins, 1991). Implementation details of the algorithm as used by industry remain proprietary except for the general outline of the approach. This article provides a design method for the upper level LP (QP) of two-stage MPC by approximating the real-time optimizer. Stability of single-stage MPC has been investigated by many authors including Garcia and Morari (1985b), Rawlings and Muske (1993), Zheng and Morari (1995), and Vuthandam et al. (1995); however, the stability of two-stage MPC has received little attention in the literature. Stability issues are significant since the MPC approach is known to fail in certain instances (Palavajjhala, 1994). Three theorems on nominal stability of two-stage MPC are given here. Robust stability is not addressed, but is considered in the application case study. Moro and Odloak (1995) seem to suggest that LP-MPC is preferred for implementing MPC in industry due to its ability to incorporate economic criteria in the control algorithm. Since single-stage MPC can also take care of some economics in the objective function, there is no convincing reason to believe that LP-MPC should outperform MPC. In this article, the economics of two-stage MPC and that of single-MPC are compared by analytic reasoning and a simulation example.

Note that as shown in Figures 1 and 2, the second-stage LP or QP is not a substitute for the real-time optimization (RTO). Rather, the two-stage MPC complements and allows the control system to track changes in the optimum caused by disturbances. The second stage uses the disturbance estimate computed by the lower-stage MPC at every sample time and then determines the optimum set point. As will be demonstrated later (using the Shell case study problem), this approach permits dynamic tracking of the optimum which is not achievable with a steady-state RTO used in conjunction with a single-stage MPC.

The analysis in the article shows that two-stage MPC is a clever way to achieve economic objectives in industrial applications. In addition it has some nice stability properties, besides guaranteeing offset free performance in the presence of step disturbances. By means of an example, it is also described why the stability region is enhanced by the LP (QP) optimizer.

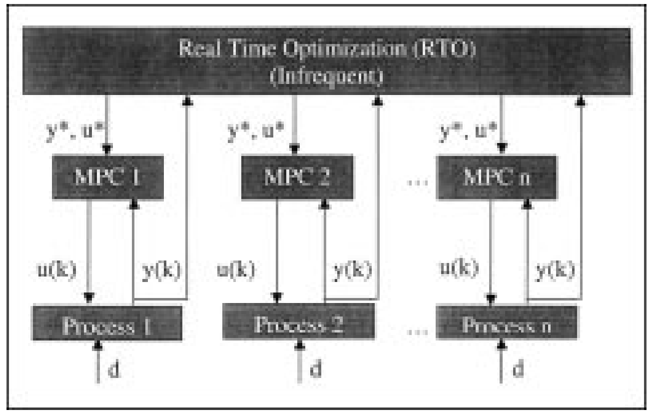

Figure 1. MPC in the control system hierarchy.

The formulation of single-stage MPC and some of its drawbacks are presented. The formulation of two-stage MPC is presented, followed by the proving of three theorems regarding the stability of two-stage MPC. The design approach of QP and LP in the second stage is presented and the economic performance of two-stage MPC is compared with that of single-stage MPC. Dynamic performance comparison is discussed. The Shell control problem is used as a test bed to verify our discussion.

Single-Stage MPC Formulation and Issues

In single-stage MPC, the controlled variables and manipulated variables have definite set points, which are calculated by a nonlinear steady-state optimizer (real-time optimizer, RTO) using an economic objective function (see Figure 1). The real-time optimizer usually works at a period of a few hours to a few days while MPC is working at a period of several seconds to several minutes.

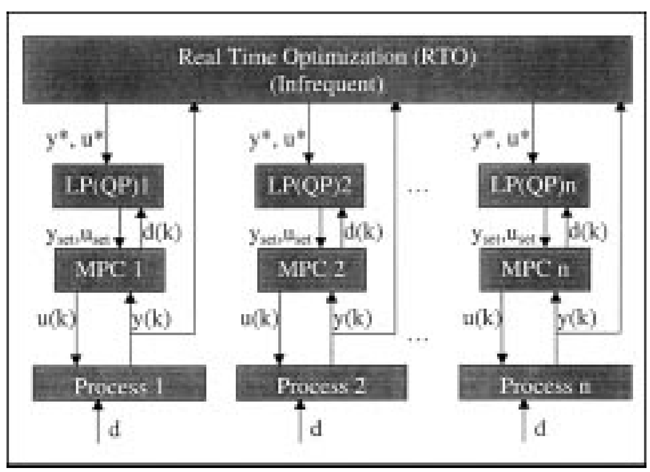

Figure 2. Two-stage MPC system hierarchy.

A general problem statement for MPC is ($y_{\text{set}} = y^*$, $u_{\text{set}} = u^*$ for single-stage MPC, with no set point change within one RTO interval)

$$\Phi_k = \min_U \sum_{j=k+1}^{k+p} (y(j|k) - y_{\text{set}})^T Q_y (y(j|k) - y_{\text{set}}) + \sum_{j=k}^{k+m-1} \left\{ [u(j|k) - u_{\text{set}}]^T R [u(j|k) - u_{\text{set}}] + \Delta u^T(j|k) S \Delta u(j|k) \right\} + \epsilon(k)^T P \epsilon(k) \quad (1)$$

subject to

$$\begin{aligned} & u_{\min} \le u(j|k) \le u_{\max}, \quad j = k, \ldots, k+m-1 \\ & y_{\min} - \epsilon(k) \le y(j|k) \le y_{\max} + \epsilon(k), \quad j = k+1, \ldots, k+p \\ & \Delta u_{\min} \le \Delta u(j|k) \le \Delta u_{\max}, \quad j = k, \ldots, k+m-1 \end{aligned} \quad (2)$$

where Q_y is a symmetric positive penalty matrix on the outputs, with y_{set} the steady-state output target and $y(j|k)$ computed from the model (an observer model can also be used)

$$\begin{aligned} & x(j+1|k) = Ax(j|k) + Bu(k), \quad j = k, \ldots, k+p \\ & y(j|k) = Cx(j|k) + \Delta(k), \quad j = k+1, \ldots, k+p \\ & \Delta(k) = y(k) - y(k|k-1) \end{aligned} \quad (3)$$

and revised by the estimated disturbance change $\Delta(k)$ which is the difference between measurements $y(k)$ and predicted output $y(k|k-1)$. R is a symmetric positive definite penalty matrix (another form of R is considered later). $u(j)$ is the input vector at time j in the open-loop objective function and u_{set} is the target for input. S is a symmetric positive semidefinite penalty matrix on the rate of change of the inputs in which $\Delta u(j|k) = u(j+1|k) - u(j|k)$ is the change in the input vector at time j. The vector U contains the m future open-loop control moves defined by

$$U = \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+m-1|k) \end{bmatrix} \quad (4)$$

for $j \ge k+m$, the input $u(j)$ is set to $u(k+m-1|k)$.

The RTO level (using a steady-state, nonlinear model of the process) gives the set points y^* and u^* for the controlled and manipulated variables. (This is used when there are extra degrees of freedom among the input manipulated variables and it is desirable to drive some of these inputs to their economic optimum.) MPC receives these set points, and its task is to move and maintain the plant as close as possible to these "optimal" set points.

The following issues arise in practical applications of MPC:

(i) When there are more input manipulated variables than controlled outputs, R can be kept nonzero to drive some inputs to desired targets. In this case even if there were no modeling errors, the presence of disturbances will cause MPC to distribute errors between y and u and reach a steady state where $y \ne y^*$. To see this, consider the situation in which the process is subject to some step disturbances. Assuming that MPC is stable, the plant will reach a new steady state at which Φ_k is minimized. Φ_k will approach a steady-state value Φ. Let (y_{ss}, u_{ss}) denote the steady state reached. Then

$$\begin{cases} y_{ss} = A_s u_{ss} + d_s \\ A_s = C(I-A)^{-1} B \end{cases} \quad (5)$$

where $d_s = \sum_{k=0}^{\infty} \Delta(k) \ne 0$ is the steady-state value of the disturbance. Assume the set points $y^* = 0$, $u^* = 0$ for convenience

$$\Phi = \sum_{j=k+1}^{k+p} y_{ss}^T Q_y y_{ss} + \sum_{j=k}^{k+m-1} u_{ss}^T R u_{ss} = u_{ss}^T \left(A_s^T p Q_y A_s + mR \right) u_{ss} + 2 d_s^T p Q_y A_s u_{ss} + d_s^T p Q_y d_s \quad (6)$$

(The input move penalty term will be zero since system is at steady state.)

For the simple case with no active constraints in the steady state

$$\frac{d\Phi}{du_{ss}} = 0 \quad (7)$$

yielding

$$u_{ss} = -\left(A_s^T p Q_y A_s + mR \right)^{-1} A_s^T p Q_y d_s \quad (8)$$

and

$$y_{ss} = \left[I - A_s \left(A_s^T p Q_y A_s + mR \right)^{-1} A_s^T p Q_y \right] d_s \quad (9)$$

where I is an $n_y \times n_y$ (n_y is dimension of output variables) unit matrix. For this to be zero (the desired set points for y)

$$A_s \left(A_s^T p Q_y A_s + mR \right)^{-1} A_s^T p Q_y = I \quad (10)$$

If $R = 0$ and the number of outputs $\le$ number of inputs (A_s has full row rank), it is guaranteed that $y_{ss} = 0$ (for case when no. of outputs < no. of inputs, there will be multiple solutions for u_{ss}, that is, system is indeterminate. Matrix $A_s^T p Q_y A_s$ is noninvertible). However, otherwise the steady-state value of y will be different from zero causing an offset.

For example, consider the following problem:

Controlled variables: y_1, y_2, y_3
Manipulated variables: u_1, u_2, u_3
Targets: $y_1 = 0$, $y_2 = 0$, $u_3 = 0$
Constraints: $y_3 \le 1$, $|u_i| \le 1$.

An MPC problem can be formulated as $\min_{u_1, u_2, u_3}(\|y_1\|^2 + \|y_2\|^2 + \|u_3\|^2)$ to achieve optimal performance. However, the presence of disturbances may drive y_3 or u_i to a constraint. In that case a degree of freedom is lost. Also, it is no longer possible to drive y_1, y_2, u_3 to their target values, leading to steady-state offsets.

(ii) Due to inevitable modeling errors and the presence of disturbances, set points y^* and u^* computed from the nonlinear steady-state model may not be consistent with the linear model used in MPC, that is, (in steady state) $y^* \neq A_s u^*$ where A_s is the steady-state gain matrix in the model used by MPC. This implies that the MPC objective of driving y to y^* and u to u^* may not be consistent (and, hence, optimal) with the constraints imposed. This will lead to steady-state offset in the implementation. In some instances y^* may even be outside the region of feasible operation defined by input constraints. This lends support for updating the set points on-line to maintain consistency and optimality.

(iii) The RTO is carried out infrequently. In between, the presence of or absence of disturbances can change the economics of operation. This is not taken into consideration by the MPC. If there are output steady-state constraints to be satisfied, the single-stage MPC must be designed to accommodate the constraints under the worst case scenario, thus making the design too conservative. Consider the example cited above with the targets changed to

$$-0.05 \leq y_1 \leq 0.05,\ -0.05 \leq y_2 \leq 0.05$$
$$\text{Economics: } \min u_3.$$

This can be achieved by redefining the control objective as $\min_{u_1, u_2, u_3} (w_1 \|y_1\|^2 + w_2 \|y_2\|^2 + w_3 \|u_3 + 1\|^2)$. By choosing w_1, w_2, w_3 appropriately, it is possible to guarantee that the outputs will remain within the specified steady-state constraints over a specified set of disturbances. However, such a design would be conservative and sacrifice economic objectives (w_3 will be small and steady state may not be on the intersections of the constraints). Such a design would also be difficult since parameters w_1, w_2, w_3 must be determined based on disturbance magnitude, as well as dynamic performance requirement.

The two-stage MPC divides the regulation problem into two levels: the upper level calculates the most economic (also consistent) set points for y and u according to the disturbance; the lower-level implements regulating action with no offsets to the set points from the upper level. Both levels are executed at the same sampling frequency (see Figure 2).

(iv) Palavajjhala (1994) reports on the application of MPC to the Tennessee Eastman problem. He observed that the large set point changes called for a RTO result in instability (MPC could not handle the sudden, large changes). This was caused by modeling errors introduced by such large changes. A gradual dynamic transition was needed to achieve a stable transition to the new optimum point. A way is needed to compute set point changes that are consistent with the underlying model used in the MPC algorithm. In this article, the Shell control problem with modeling error is used to illustrate the advantage of LP-MPC in consistent set point transfer, as well as to demonstrate some of the robustness advantages of LP-MPC.

Two-Stage LP-MPC (QP-MPC) Cascade Control Strategy

The two-stage MPC inserts an optimization layer between MPC and RTO level, as shown in Figure 2. This layer provides manipulated and controlled variable set points to the lower-level MPC.

The real-time optimizer infrequently updates the optimal nominal "targets" y^*, u^* and the cost parameters guiding the LP (or QP). It can also update the constraints for LP (QP) if necessary. The RTO is based on nonlinear steady-state models. The LP (QP) is executed at the same frequency as lower stage MPC.

Consider the following LP formulation (Morshedi et al., 1985; Brosilow and Zhao, 1988; Yousfi and Tournier, 1991; Harkins, 1991)

$$\min_{y_{\text{set}}, u_{\text{set}}} c_y^T (y_{\text{set}} - y^*) + c_u^T (u_{\text{set}} - u^*) + c_\epsilon^T \epsilon \tag{11}$$

s.t.

$$\begin{aligned} & y_{\text{set}} = A_s u_{\text{set}} + d(k) \\ & d(k) = d(k-1) + \Delta(k) \\ & y_{\min} - \epsilon \leq y_{\text{set}} \leq y_{\max} + \epsilon \\ & u_{\min} \leq u_{\text{set}} \leq u_{\max} \\ & A_s = C(I - A)^{-1} B \\ & \epsilon \geq 0 \end{aligned} \tag{12}$$

where $d(k)$ is the estimated disturbance at time k. c_y, c_u are cost parameters (see the section on how these are updated by the real-time optimizer). ϵ is used to guarantee a feasible solution to the LP. c_ϵ is a tuning parameter. Strict specifications on outputs (such as purity specifications) can be included. The presence of $d(k)$ (disturbance estimate) makes this a dynamically changing problem. This feedback term also raises the question of stability of the LP-MPC cascade scheme.

Similarly, a QP problem may be setup as (Muske and Rawlings, 1993)

$$\min_{y_{\text{set}}, u_{\text{set}}} (y_{\text{set}} - y^*)^T C_y (y_{\text{set}} - y^*) + (u_{\text{set}} - u^*)^T C_u (u_{\text{set}} - u^*) + c_y (y_{\text{set}} - y^*) + c_u (u_{\text{set}} - u^*) + \epsilon^T c_\epsilon^T c_\epsilon \epsilon \tag{13}$$

with the same constraints as those in LP problem (C_y, C_u are also parameters from the real-time optimizer). If $C_y = C_u = 0$, then QP degrades to LP. An approach will be given on how the initial weights are calculated from the RTO problem setup.

Some theorems are discussed first, which relate to the stability properties of the LP(QP)-MPC algorithms. Theorem 1 addresses the stability of the two-stage algorithm in the absence of constraints. Theorem 2 considers the effect of constraints at the LP (QP) level and Theorem 3 addresses stability when constraints are present at both levels.

Stability Theorems for Two-Stage MPC

The following theorem addresses stability when no constraints are present.

Theorem 1 (Unconstrained Case). If the system model is perfect, and the upper stage of two-stage MPC is an unconstrained QP, while the lower stage is an (unconstrained) DMC controller, then the QP-DMC controlled system is stable provided the DMC controlled system is stable.

Proof. Consider the lower-level DMC controller. Without constraints, this is a *linear* controller and, hence, the transfer function of the closed-loop can be explicitly derived (Garcia and Morari, 1985a). When the top level QP (without any active constraints) is used to determine y_{set}, u_{set}, the transfer function will change. Since the model in the lower level MPC is perfect, the estimated disturbance at any time will be exact. The objective function of top level QP formulation is

$$\min_{y_{\text{set}}, u_{\text{set}}} \left[A_s u_{\text{set}} + d(k) - y^* \right]^T c_y^T c_y \left[A_s u_{\text{set}} + d(k) - y^* \right] + (u_{\text{set}} - u^*)^T c_u^T c_u (u_{\text{set}} - u^*) \quad (14)$$

The optimal y_{set}, u_{set} are determined by $d(k)$ and are given by

$$u_{\text{set}} = \left(A_s^T c_y^T c_y A_s + c_u^T c_u \right)^{-1} \left[A_s^T c_y^T c_y (y^* - d(k)) + c_u^T c_u u^* \right]$$
$$y_{\text{set}} = A_s u_{\text{set}} + d(k) \quad (15)$$

Since all the matrices are constant, the above expression can be rewritten

$$\begin{aligned} u_{\text{set}} &= f_1 y^* + f_2 u^* - f_1 d(k) \\ y_{\text{set}} &= g_1 y^* + g_2 u^* + g_3 d(k) \end{aligned} \quad (16)$$

where

$$f_1 = \left(A_s^T c_y^T c_y A_s + c_u^T c_u \right)^{-1} A_s^T c_y^T c_y,$$
$$f_2 = \left(A_s^T c_y^T c_y A_s + c_u^T c_u \right)^{-1} c_u^T c_u \quad (17)$$
$$g_1 = A_s f_1, \qquad g_2 = A_s f_2, \qquad g_3 = I - g_1. \quad (18)$$

f_1, f_2, g_1, g_2, g_3 are constant matrices. As can be seen, the top layer is nothing but a feedforward controller. Hence, if the lower level DMC is stable, then QP-DMC must also be stable.

With the prior knowledge of all these constant matrices, a comparison between the performances of DMC and QP-DMC could be obtained.

The second theorem concerns the stability when constraints are introduced at the upper level LP or QP.

Theorem 2 (Constrained LP or QP). If the system model is perfect, and the upper two-stage MPC is a constrained QP (or LP) with the same active (linear) constraints all the time, while the lower is an (unconstrained) DMC controller, then the QP(LP)-DMC controlled system is stable provided the DMC controlled system is stable.

Proof. When the active constraints do not change, y_{set}, u_{set} are at the intersection of these constraints and are linear functions of d, y^*, u^* only. Thus, the top layer is still a feedforward controller. Hence, stability holds for QP(LP)-DMC control system also.

These two theorems suggest that a DMC controller can be tuned first, then a LP (QP) layer can be added on the top without affecting the stability of the control system. If the top layer improves the performance of the control system, it is kept.

If constraints are present at the lower-level MPC algorithm, there is a similar stability theorem.

Theorem 3 (Constrained MPC). If the plant (A) is stable, the model is perfect and the lower part of a two-stage MPC is a controller with $m \geq 1$, $P < \infty$ and $p = \infty$ in Eq. 1, and the controller output is kept satisfying an end condition at the end of the control horizon

$$u(j|k) = u_{\text{set}}, \quad j = k + m, \quad k + m + 1, \ldots, \infty \quad (19)$$

then the system is globally asymptotically stable provided that the disturbance is a constant after some finite time. Also, $y(k) = y_{\text{set}} =$ constant is an asymptotically stable solution of the closed-loop two-stage MPC control system.

Proof. Since the disturbance is constant after some finite time, the set points passed from the upper stage will not change either after a finite time, say k_d. After time k_d, the lower-level MPC will be in the format of Eq. 1 with $m \geq 1$, $P < \infty$, $p = \infty$, and fixed set points. Since k_d is finite, the state and output variables at that time must be finite, and after k_d, the problem is equal to a single-stage MPC problem whose stability has been proved by Rawlings and Muske (1993) and Zheng and Morari (1995). The objective function $\Phi(k)$ in Eq. 1 is decreasing and is bounded by zero

$$\left[y(k) - y_{\text{set}} \right]^T Q_y \left[y(k) - y_{\text{set}} \right] + \left[u(k) - u_{\text{set}} \right]^T R \left[u(k) - u_{\text{set}} \right] + \Delta u^T(k) S \Delta u(k) + \Phi(k+1) \leq \Phi(k) \leq \Phi(k_d) < \infty \quad (20)$$

Thus, $u(k) = u_{\text{set}}$, $y(k) = y_{\text{set}}$ at $k \to \infty$ because $Q_y > 0$, $R > 0$, $S \geq 0$ (these are stated in Eq. 1). The two-stage MPC system is stable.

Remark 1. If in Eq. 1 $Q_y \geq 0$ (positive semidefinite), the above theorem still holds. Since $R > 0$, $u(k) = u_{\text{set}}$, $y(k) = A_s u(k) + d(k) = y_{\text{set}}$ at $k \to \infty$.

Remark 2. If in Eq. 1 $R \geq 0$, and the static gain matrix A_s in Eq. 5 has full column rank, theorem 3 is still valid.

Since the static gain matrix A_s in Eq. 5 has full column rank, that means when $y(k) = y_{\text{set}}$, $k \to \infty$, there is one and only one state for $u(k)$, $k \to \infty$ which is u_{set}.

The above theorems suggest that adding an optimization layer can dynamically track the optimum while maintaining the system's stability. The assumptions used are mild, and as is demonstrated below in a case study, the stability is improved rather than decreased by adding the second layer.

For single-stage MPC under step disturbances, since the set point is not changing according to disturbances, the objective function will become infinite if infinite horizon is used and the problem is not defined. Therefore, only a finite horizon formulation exists. Vuthandam et al. (1995) provided a theorem of stability which requires the end condition constraints for single-stage MPC (EQDMC)

$$u(j|k) = A_s^{-1}\left(y_{\text{set}} - d(k) \right), \quad j = k + m, \quad k + m + 1, \ldots, \infty \quad (21)$$

Similar stability theorem could be developed for LP-EQDMC or QP-EQDMC. However, the addition of the end constraints may shrink the feasible region of MPC (Lee and Cooley, 1997). This can be illustrated with an extreme example. Suppose the allowable change on input is so small that at predicted time step $k+m-1$, the input is still far from the steady-state value. In this case, if the end constraints are added, the required change of input at time $k+m-1$ may be larger than the maximum allowable change of input, causing the problem to be unsolvable if no hard constraint violation is allowed.

Design Approach and Steady-State Properties of Two-Stage LP(QP)-MPC Algorithm

First, RTO is considered. Typically, this is a nonlinear programming problem (NLP) of the form

$$\min_{y,u} f(x,u) \tag{22}$$

subject to

$$g(x,u) \geq 0 \tag{23}$$

$$h(x,u) = 0 \tag{24}$$

with plant output

$$y = K(x,u) \tag{25}$$

The solution to this problem can be said to be given at the point (x^*, u^*, y^*).

Single-stage MPC implements (y^*, u^*) and drives the plant to this steady-state optimum. However, this may not be the true optimum of the actual plant due to the presence of modeling errors and the presence of external disturbances. Two-stage MPC tries to follow the changes in optimum operating conditions of a process unit caused by the presence of external disturbances. (Modeling error is treated the same way as an external disturbance.)

Design of second-stage QP in QP-MPC

To construct the second-stage QP problem, a quadratic approximation to the RTO NLP is considered first, following the logic used in successive quadratic programming (SQP) methods used for solving NLP problems.

Let

$$z = \begin{bmatrix} x \\ u \end{bmatrix} \tag{26}$$

Then the RTO problem becomes

$$\min_{y,u} f(z) \tag{27}$$

subject to

$$g(z) \geq 0 \tag{28}$$

$$h(z) = 0 \tag{29}$$

Define a Lagrangian function

$$L(z,\upsilon,w) = f(z) - \sum \upsilon_k h_k(z) - \sum w_j g_j(z) \tag{30}$$

where υ, w are Lagrangian variables. Given an optimum solution to the problem at (z^*, υ^*, w^*), a QP approximation can be constructed for the RTO at (z^*, υ^*, w^*)

$$\min_{z_d} \nabla f(z^*)^T z_d + \frac{1}{2} z_d^T \nabla_z L(z^*, \upsilon^*, w^*) z_d \tag{31}$$

subject to

$$g(z^*) + \nabla g(z^*)^T z_d \geq 0 \tag{32}$$

$$h(z^*) + \nabla h(z^*)^T z_d = 0 \tag{33}$$

Han (1976) and Powell (1978) showed that, if the solution to the NLP (x^*, u^*) satisfies the Kuhh-Tucker conditions of optimality, then this solution will also satisfy the conditions of optimality of the QP approximation. This analysis forms the basis of the QP used in two-stage MPC. One starts with the solution $z^* = (x^*, u^*)$. If there are no modeling errors and there are no disturbances, then this QP will yield the same solution. When a disturbance enters the system, it affects the constraints used and the optimal solution will shift. The objective of the two-stage MPC is to try and follow this shifting optimum operating point of the plant.

Some of the variables can be eliminated in Eqs. 31, 32, 33, and the QP can be reduced in terms of u and y only. In this case the problem reduces to (posing it using deviations from y^*, u^* and including disturbance effects explicitly)

$$\min_{y_{\text{set}}, u_{\text{set}}} (y_{\text{set}} - y^*)^T C_y (y_{\text{set}} - y^*) + (u_{\text{set}} - u^*)^T C_u (u_{\text{set}} - u^*) + c_y(y_{\text{set}} - y^*) + c_u(u_{\text{set}} - u^*) \tag{34}$$

subject to

$$\begin{aligned} &y_{\text{set}} = A_s u_{\text{set}} + d(k) \\ &d(k) = d(k-1) + \Delta(k) \\ &y_{\min} \leq y_{\text{set}} \leq y_{\max} \\ &u_{\min} \leq u_{\text{set}} \leq u_{\max} \end{aligned} \tag{35}$$

where C_y, C_u, c_y, and c_u can be derived from Eq. 31. Note that a number of other changes were made also in the QP. First, the linear relationship between y and u is replaced with the actual linear model employed in the MPC. The linearization of the steady-state model is usually not accurate enough. Secondly, the inequality constraints have been replaced with bounds on y and u only. If necessary, other linear inequality constraints can be included relating y and u as well. The former is based on constraints employed at the MPC level.

This formulation takes care of the two issues raised earlier regarding model errors and disturbances. The disturbance $d(k)$ is updated to reflect the current measurements available on the plant and, hence, the relationship used matches the plant data

$$\begin{aligned} &d(k) = d(k-1) + \Delta(k) \\ &\Delta(k) = y(k) - y(k|k-1). \end{aligned} \tag{36}$$

We now have a QP that is consistent with the plant data. The solution obtained $(y_{\text{set}}, u_{\text{set}})$ is thus reachable by the lower-level MPC. (Recall that (y^*, u^*) may form an unreachable set.)

There is one additional consideration. The QP problem as above may not have a feasible solution. It may be desirable to soften the output constraints using a penalty variable ϵ

$$\min_{y_{\text{set}}, u_{\text{set}}} (y_{\text{set}}-y^*)^T C_y (y_{\text{set}}-y^*) + (u_{\text{set}}-u^*)^T C_u (u_{\text{set}}-u^*) + c_y (y_{\text{set}}-y^*) + c_u (u_{\text{set}}-u^*) + \epsilon^T c_\epsilon^T c_\epsilon \epsilon \quad (37)$$

Soften the output constraints with

$$\begin{gathered} y_{\min} - \epsilon \leq y_{\text{set}} \leq y_{\max} + \epsilon \\ \epsilon \geq 0 \end{gathered} \quad (38)$$

c_ϵ forms a tuning parameter.

The corrected set points $(y_{\text{set}}, u_{\text{set}})$ are passed onto the lower-level MPC. The following is an important property of the two-stage MPC.

No steady-state offset property

In the presence of stable (step-wise) disturbances, the two-stage MPC will reach a steady state with no offsets ($y_{ss} = y_{\text{set}}$, $u_{ss} = u_{\text{set}}$).

Analysis. When the disturbance reaches its final value, the following equation will be valid (for both the model as well as the plant)

$$y_{\text{set}} = A_s u_{\text{set}} + d_s \quad (39)$$

because of the constraints in the upper stage. d_s accounts for the modeling error, as well as the steady-state disturbance.

If the objective function in the upper level does not change, the set points passed to the lower level will remain unchanged after the disturbance estimation reaches steady state. Then, at the lower-level MPC, the steady-state objective function will become

$$\Phi = \sum_{j=k+1}^{k+p} (y_{ss} - A_s u_{\text{set}} - d_s)^T Q_y (y_{ss} - A_s u_{\text{set}} - d_s) + \sum_{j=k}^{k+m-1} (u_{ss} - u_{\text{set}})^T R (u_{ss} - u_{\text{set}}). \quad (40)$$

Since $\Phi \geq 0$ and $y_{ss} = A_s u_{ss} + d_s$ is also satisfied, the solution to this optimization problem is $y_{ss} = A_s u_{\text{set}} + d_s = y_{\text{set}}$ and $u_{ss} = u_{\text{set}}$ with the resulting objective function zero. Therefore, for two-stage MPC, no offset will appear as long as the control system is stable.

Design of second-stage LP in LP-MPC formulation

Instead of using QP, a linear approximation of the RTO objective function is also employed. The LP approximation is given by

$$\min_{z_d} \nabla f(z^*)^T z_d \quad (41)$$

subject to

$$g(z^*) + \nabla g(z^*)^T z_d \geq 0 \quad (42)$$

$$h(z^*) + \nabla h(z^*)^T z_d = 0 \quad (43)$$

If there are output variables given by

$$y = K(z) = K(x, u), \quad (44)$$

they can be linearly approximated as

$$y = K(x^*, u^*) + \nabla_x K(x^*, u^*)^T (x - x^*) + \nabla_u K(x^*, u^*)^T (u - u^*) \quad (45)$$

Like in QP design, this may be simplified and cast in terms of deviation variables $(y_{\text{set}} - y^*)$ and $(u_{\text{set}} - u^*)$ as follows

$$\min_{y_{\text{set}}, u_{\text{set}}} c_y^T (y_{\text{set}} - y^*) + c_u^T (u_{\text{set}} - u^*) \quad (46)$$

with the constraints

$$\begin{aligned} & y_{\text{set}} = A_s u_{\text{set}} + d(k) \\ & d(k) = d(k-1) + \Delta(k) \\ & y_{\min} \leq y_{\text{set}} \leq y_{\max} \\ & u_{\min} \leq u_{\text{set}} \leq u_{\max} \end{aligned} \quad (47)$$

Output constraints can be softened by adding a penalty term

$$\begin{gathered} y_{\min} - \epsilon \leq y_{\text{set}} \leq y_{\max} + \epsilon \\ \epsilon \geq 0 \end{gathered} \quad (48)$$

The objective function now becomes

$$\min_{y_{\text{set}}, u_{\text{set}}} c_y^T (y_{\text{set}} - y^*) + c_u^T (u_{\text{set}} - u^*) + c_\epsilon^T \epsilon \quad (49)$$

In practice, the cost objective function may also be set up based on heuristic or intuitive arguments. For example, if the outputs are strictly constrained (such as purity specifications), then most of the economics is derived from minimizing the cost of inputs. In this case $c_y = 0$, $c_u =$ cost associated with each input manipulated variable.

Steady-State Economic Performance

If the error introduced by the QP (or LP) approximation to the nonlinear RTO objective function is small, then the two-stage MPC will yield better steady-state economic performance than single-stage MPC when the plant is subject to constant disturbances.

The presence of sustained (constant) disturbances shifts the economics optimum from the original (y^*, u^*) computed by the real-time optimizer. Since the real-time optimizer is not invoked frequently, the process may operate at suboptimal conditions under single-stage MPC. The two-stage MPC, on the other hand, utilizes on-line measurement feedback to continuously update the disturbance effect on the output. This, in turn, allows the LP or QP (approximation of the RTO) to move to a more economic set of operating conditions (y_{set}, u_{set}) than those reached by single-stage MPC.

Three cases are considered.

Case 1. If strict equality constraints are imposed on the output variables (such as strict purity or composition specifications) in the form

$$y^* \le y_{\text{set}} \le y^* \tag{50}$$

then some degrees of freedom are lost, and there may be no advantage to use a two-stage MPC. For the case where all outputs are strictly constrained and the number of inputs is equal to the number of outputs, the steady state reached by the single-stage MPC and two-stage MPC are the same independent of the disturbances.

Case 2. If the number of inputs > the number of outputs, there will be some degree of freedom left in the QP(LP) to achieve a more economic set of operating conditions u_{set}.

Case 3. The most benefit of the two-stage MPC occurs when the output variables are not strictly constrained, that is

$$y^* - \Delta y_{ss} \le y_{\text{set}} \le y^* + \Delta y_{ss}, \tag{51}$$

where Δy_{ss} represents the allowable variation in the output variables.

This allows the most degree of freedom for the QP(LP) and, hence, there is greater opportunity to optimize.

In most practical cases, the optimum lies at the intersection of the constraints. The presence of sustained disturbances can shift the optimum from one intersection to another. The QP-MPC and LP-MPC scheme is able to detect and execute this move as disturbances come in, thus allowing superior economic performance. Since the set points from LP(QP) are reachable, LP-MPC or QP-MPC will stabilize on the intersection of steady-state constraints. If LP or QP is a good approximation of the RTO, the steady state reached by LP-MPC or QP-MPC will be very close to the real economic optimum.

It is possible to retain some economic objective in the formulation of the single-stage MPC for some special cases. Consider the case where economic penalty is associated with input variable deviation from u^*. In this case, the following objective function can be used for the single-stage MPC

$$\begin{aligned}\Phi_k = \min_U & \sum_{j=k+1}^{k+p} [y(j|k) - y^*]^T Q_y [y(j|k) - y] \\ & + \sum_{j=k}^{k+m-1} \{[u(j|k) - u^*]^T R[u(j|k) - u^*] \\ & + \Delta u^T(j|k) S \Delta u(j|k)\} + \epsilon(k)^T P \epsilon(k)\end{aligned} \tag{52}$$

By choosing Q_y and R wisely, one can attempt to make sure that the steady-state offsets are within the specification

$$y^* - \Delta y_{ss} \le y_{ss} \le y^* + \Delta y_{ss} \tag{53}$$

Alternatively, Eq. 53 can be incorporated as an end point constraint in the MPC formulation. For the former method, the steady state reached may not be at the intersection of constraints. Thus, it's not likely to be economically optimal. Besides, the choice of Q_y and R is a difficult tuning problem. For the end point constraints method (EQDMC), the end point constraints will cause the MPC problem to be too restrictive (Lee and Cooley, 1997), that is, MPC problem may be insolvable.

Finally, if the RTO problem can be solved as frequently as LP or QP, there is no need to adopt the two-stage MPC strategy. Single-stage MPC should work fine since the RTO is updating its set points continuously and frequently. However, if the RTO is a large-scale NLP problem, then it may not be possible to solve it at the same frequency as the MPC.

Dynamic Performance Analysis

A qualitative look is now taken at the dynamic performance of the two-stage MPC algorithm. Consider the MPC formulation of Eq. 1 with set point zero. If a step disturbance comes in at some time, for single-stage MPC, large changes on the input are suppressed because of the nature of the objective function. Therefore, the controller action will be sluggish. This can cause larger deviations on the output response and longer settling down times. In some cases, it may cause the quadratic problem to be without a feasible solution if hard output constraints are present.

Now consider the two-stage MPC under the same conditions. If a step disturbance comes in, the upper level will detect the disturbance and adjust the set point of u accordingly. The corrected set point of u will immediately be delivered to the second-stage MPC. This immediate correction of the input set points will allow larger input change and keep the output deviations smaller. The control action will become smaller as the process approaches the new set points, allowing smooth settling of the process.

The two-stage MPC is also beneficial when considering a global set point transfer. This can be illustrated by the following example.

Suppose there are two sequential processes: a reactor subsystem and a separation subsystem (see Figure 3). Assuming that the system is controlled by an MPC (or LP-MPC) controller with a top level RTO determining the set points, when

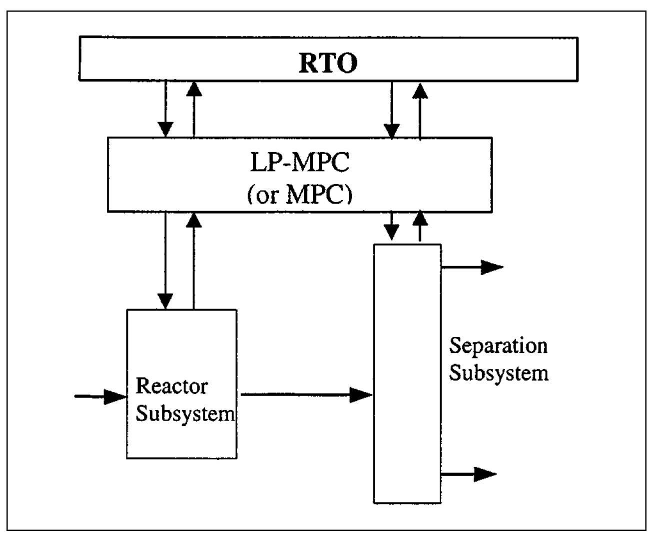

Figure 3. Reactor-separation system.

processing conditions change, the RTO will give new set points for both the reactor subsystems and the separation subsystem. The MPC controller for the separation subsystem will try to move the controlled variables to the new set points, assuming reactor subsystem output has reached new levels. However, because of the delay and lag in the reactor, the reactor output will lag behind. Under some conditions, the new set points in the separation subsystem may not even be reachable until the reactor output has reached the new set point. Thus, moving of the reactor and separation subsystems to the new global steady-state optimum must be done keeping in mind the dynamic lags present in the system.

In the case of the two-stage MPC, the real-time optimizer can transfer the new target values for the input and output along with their economic significance (C_y, C_u, c_y, c_u) to LP or QP. The LP or QP can in turn compute new set points for the lower-level MPC controller, which are consistent with the current plant operating conditions.

LP vs. QP at the top level

Both LP and QP optimize the process subject to constraints. The main difference is that the LP solution will be at the intersection of constraints. As disturbances enter the process, the optimum may shift. In the case of LP, this may result in a jump from one intersection to the other. This could cause the set point on y and u to change abruptly, which is detrimental to the stability of the second-stage MPC. With QP, however, this kind of situation is less likely to occur. The control quality of the whole system, thus, may be better.

Another drawback of using LP is that the LP problem may have multiple solutions (such as, the LP solution may be along one side of the feasible solution polygon). In such a case, one has to design an appropriate approach in order to select a solution which will be passed to the lower-level MPC from the solution set. QP, however, will have a unique solution.

In case disturbances are varying rapidly in time (that is, dynamics of the disturbances are important), it may be desirable to add a filter to slow down the set point changes in the lower stage.

Shell Control Problem

The Shell control problem (Prett and Morari, 1986) is a multivariable problem concerning control of a heavy oil fractionator. The column setup is shown in Figure 4, and the transfer functions are tabulated in Appendix.

The control objectives and constraints are stated as follows:

(a) Regulatory objective:

$$-0.005 \le y_1, \quad y_2 \le 0.005 \text{ at steady state}$$

(b) Economic objective:

$$u_3 \text{ should be minimized}$$

(c) Output constraints:

$$-0.5 \le y_1 \le 0.5$$
$$y_7 \ge -0.5$$

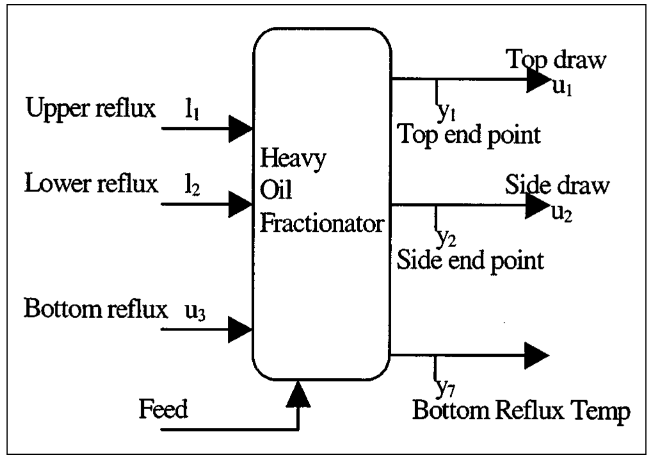

Figure 4. Shell heavy oil fractionator.

(d) Manipulated variable constraints:

$$-0.5 \le u_i \le 0.5, \quad i = 1,2,3$$
$$|\Delta u_i| \le 0.05/\text{min}$$

(e) Disturbances l_1, l_2 are unmeasured.

There are three output and three input variables namely: y_1, y_2, y_7 and u_1, u_2, u_3. Since it is needed only to keep y_7 above -0.5, there is one degree of freedom on the manipulated variable (unless one is up against some constraint). The economic objective is to minimize use of u_3. One way to achieve this is the single-stage MPC formulation

$$\underset{\substack{u_{1,k+j}, u_{2,k+j}, u_{3,k+j} \\ j=0,\ldots,m-1}}{\text{Min}} \sum_{i=1}^{p} \left[w_y (y_{1,k+i} - y_{1\text{set}})^2 + w_y (y_{2,k+i} - y_{2\text{set}})^2 \right] + \sum_{j=0}^{m-1} w_u (u_{3,k+j} - u_{3\text{set}})^2 \quad (54)$$

subject to

$$\begin{aligned}
&-0.5 \le y_{1,k+j} \le 0.5, \quad 1 \le j \le p \\
&-0.5 \le y_{7,k+j}, \quad 1 \le j \le p \\
&-0.5 \le u_{1,k+j} \le 0.5, \quad 0 \le j \le m-1 \\
&-0.5 \le u_{2,k+j} \le 0.5, \quad 0 \le j \le m-1 \\
&-0.5 \le u_{3,k+j} \le 0.5, \quad 0 \le j \le m-1 \\
&-0.05 \cdot t_s \le \Delta u_{1,k+j} \le 0.05 \cdot t_s, \quad 0 \le j \le m-1 \\
&-0.05 \cdot t_s \le \Delta u_{2,k+j} \le 0.05 \cdot t_s, \quad 0 \le j \le m-1 \\
&-0.05 \cdot t_s \le \Delta u_{3,k+j} \le 0.05 \cdot t_s, \quad 0 \le j \le m-1
\end{aligned} \quad (55)$$

where t_s is the sample time. Here weights are put on y_1 and y_2, since it is assumed that they are equally important. By adjusting the relative values of w_y and w_u, more or less weight can be given to economics vs. regulation. (The assumption in single-stage MPC is that the real-time optimizer gives out y_1^*, y_2^*, and u_3^* calculated from disturbances at the time when the real-time optimizer is executed.) These targets may not be optimal when disturbances change later.

Many researchers have tried to solve the Shell problem using different approaches as described in the monograph by Prett et al. (1990). Cuthrell et al. (1988) proposed using single-stage MPC (QDMC) to solve this problem. However, they did not include any economic objectives in their MPC algorithm. The performance of the nonlinear programming (NLP) method they propose is similar to that of the QDMC. Economics is not considered in this NLP formulation either. Wang et al. (1988) introduced the LQG/LTR robust control design methodology, which included the economics in the objective function. However, they did not consider any modeling errors in the examples studied. Besides, steady-state errors are present when a step disturbance is introduced. Holt and Lu (1988) used the so-called scheduled controller, which minimizes the worst case performance for a nonlinear system with parameter uncertainty. Basically, their approach, if solved on-line, is analogous to the approach used in QDMC. Rawlings and Eaton (1988) proposed the QDMC algorithm plus feedforward control for the load rejection in the Shell control problem. First, the targets of manipulated variables are determined through the steady-state model with measured load. Then, the targets are passed to MPC for control. Although they include economics in the objective function and their approach works better than QDMC without feedforward, the effectiveness of the algorithm depends on the availability of the load measurement. McDonald and Palazoglu (1988) presented a multiobjective predictive control solution. However, the output constraints are violated in some cases. In the monograph of fundamental process control (Prett and Garcia, 1988), the Shell problem was the major test bed for the unconstrained DMC and constrained DMC (QDMC). Although economic performance was considered, the combination of multiple objectives into one objective does not allow the designer to reflect the true performance requirements. Zafiriou (1990) reexamined the robustness of QDMC strategy. Yu et al. (1994) used the so-called state estimation based model predictive control (SEMPC). Maciejowski (1994) proposed the multivariable Smith predictive control and applied it to the Shell problem. Others (Vuthandam et al., 1995) used QDMC with end condition (EQDMC).

Here, the use of single-stage MPC (QDMC), as well as two-stage MPC to control this system, is compared. The comparisons show that two-stage MPC (LP-MPC and QP-MPC) performs better than single-stage MPC. A finite horizon formulation is used in these comparisons. (Infinite horizon cannot be used for single-stage MPC since the objective function would become infinite.)

All MPC computations were performed using the MPC toolbox in Matlab. The parameters used were: Control move horizon = 2; Prediction horizon = 30; Move suppression factor = 0; and Sample time = 6 min.

The LP objective function adopted here is

$$\underset{u_{1,\mathrm{set}},\,u_{2,\mathrm{set}},\,u_{3,\mathrm{set}}}{\mathrm{Min}} \quad (u_{3,\mathrm{set}}) \qquad (56)$$

subject to the steady-state constraints

$$\begin{aligned}
&-0.005 \le y_{1,\mathrm{set}} \le 0.005\\
&-0.005 \le y_{2,\mathrm{set}} \le 0.005\\
&-0.5 \le y_{7,\mathrm{set}}\\
&-0.5 \le u_{\mathrm{set}} \le 0.5\\
&y_{\mathrm{set}} = A_s u_{\mathrm{set}} + d(k)\\
&d(k) = d(k-1) + \Delta(k)
\end{aligned} \qquad (57)$$

(i) to (iv) below assume that the model is perfect. Plant modeling errors are considered in (v) and (vi). Disturbances are unmeasured. The RTO is the same as second-stage LP, except it is executed at a less frequent manner.

The above setup of MPC may not be optimal. However, since LP-MPC is using the same lower-level MPC setup, the comparison is fair. The following points on the performances of both single-stage MPC and LP-MPC are compared.

(i) Dynamic tracking of economic optimum with LP-MPC

For single-stage MPC ($u_{3\mathrm{set}} = u_3^*$), the term $(u_{3,k+j} - u_{3\mathrm{set}})^2$ in the objective function, that is, $w_y = 2$, $w_u = 1$, is included. The initial disturbance is $l_1 = l_2 = 0.4$, from which the real-time optimizer gives out the consistent set point $u_3^* = -0.24$. However, it fails to track the economic optimum on u_3 as disturbances change to $l_1 = l_2 = 0$ later. For LP-MPC, with the same MPC setting, it tracks the optimum of u_3 dynamically and Figure 5 shows that LP-MPC is running at a more economic state after $t = 120$. Note that, for LP-MPC, some optimality is sacrificed to get better regulatory performance in the initial part when the disturbances enter the process. Over the long run, the payback from the LP-MPC optimization can be substantial.

(ii) Steady-state performance

Table 1 gives the results of steady-state offsets for both MPC and LP-MPC. The weights in the objective function of

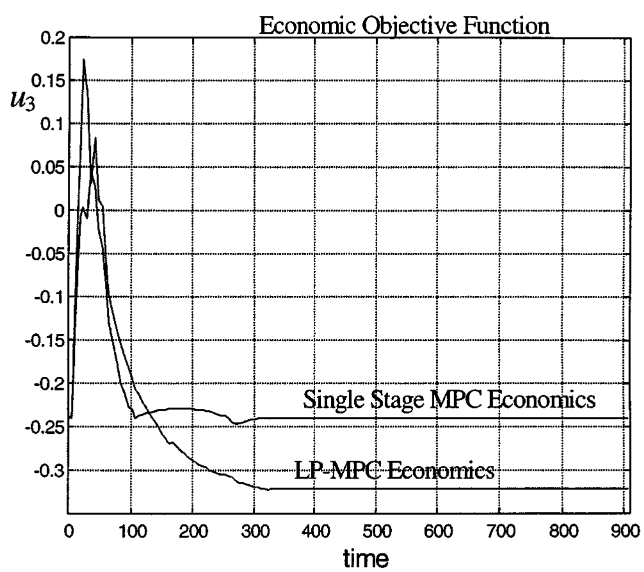

Figure 5. Ability shown of LP-MPC to track the optimum dynamically.

Response of u_3 to step disturbance changes (from $l_1 = l_2 = 0.4$ to $l_1 = l_2 = 0$). Objective is to minimize u_3 (maximizes steam make).

Table 1. Steady States Reached by Control Systems when Subject to Step Disturbance Change from $l_1 = l_2 = 0$ to $l_1 = l_2 = 0.4$

Variable	Set Point Constraint	Actual SS Reached by LP-MPC	Actual SS Reached by MPC
Top end point, y_1	0 ± 0.005	0.005	0.0052
Side end point, y_2	0 ± 0.005	0.005	0.0187
Economic objective, u_3	—	−0.2401	−0.2471

Eq. 54 are $w_y = 2$, $w_u = 1$ for both cases. It's seen that single-stage MPC leads to a violation of specifications on y_1 and y_2 due to the inclusion of economics in the objective function. LP-MPC effectively maintains the specification while achieving optimal economic performance (steady state at the intersection of the constraints).

A well-formulated single-stage MPC can also maintain the output steady state in the specified region, but at the cost of sacrificing economic performance and/or stability. Generally, there are two ways, specifically:

(a) Increase the weight on controlled variables w_y so that even in the worst case, steady-state specification could be met. For example, making $w_y = 10$, $w_u = 1$ will reach a steady state of $y_{1ss} = 0.0001$, $y_{2ss} = 0.0002$, $u_{3ss} = -0.2379$. However, this is not economically optimal since the optimal solution is on the boundary of the feasible region. The final value reached u_{3ss} is larger than reached by LP-MPC (−0.2401). Hence, even though the steady-state specification is met, one loses economics by doing so.

(b) Incorporate hard constraints to MPC which bound the controlled variables at the end of the control horizon within the specified steady-state feasible region (that is, $-0.005 \le y_{1ss}, y_{2ss} \le 0.005$). However, this will reduce the feasible region of the MPC problem as seen in point (iii), and the ability to handle disturbances is jeopardized. If one chooses to soften the hard constraints, then steady-state specifications may not be met.

Thus, there is a tradeoff for single-stage MPC between the steady-state specifications and economic performance.

(iii) Disturbance handling ability

Since the strict output constraints stated in the problem (Eq. 55) have been enforced, there are limits to the amplitude of disturbances before the controllers fail (no feasible region of MPC problem). Table 2 lists the allowable range of disturbance for both algorithms. The weights in Eq. 54 for both control strategies are again the same $w_y = 2$, $w_u = 1$ for fair comparison. When the same amplitude of disturbances come in, the single-stage MPC will have larger overshoot than LP-MPC. This in turn favors LP-MPC in disturbance handling ability. Also shown in Table 2 is the maximum disturbance amplitude that can be handled by EQDMC. Since there are extra constraints in EQDMC, the disturbance handling ability is further reduced.

Table 2. Comparison of Largest Amplitude of Disturbances LP-MPC and Single-Stage MPC Can Handle ($l_1 = l_2$) with Initially No Disturbance

Method	Max. Δu	Max. Endurable Step Change in Disturbance (l_1, l_2)
LP-MPC	0.05/min	0.64
MPC	0.05/min	0.43
EQDMC	0.05/min	0.33

(iv) Dynamic performance

The responses to disturbances for both cases are also plotted in Figure 6 (weights are still the same $w_y = 2$, $w_u = 1$). The LP-MPC performs better. The reason that LP-MPC performs slightly better is due to its ability to make larger moves in u_3 because of set point update. Since u_3 is included in the economic objective in the case of single-stage MPC, its movements are slightly more constrained and this causes the output to deviate from the target a little more than the LP-MPC algorithm.

(v) Global set point transfer

Sudden large set point changes may cause instability for MPC controllers. This is tempered by constraints imposed on inputs and move sizes. With some changes on the statement of Shell problem, the advantage of the two-stage cascade MPC in making global set point transfer can be illustrated. Here model mismatch is introduced so that one can evaluate the robustness of LP-MPC, as well as MPC.

The objective functions of both MPC (Eq. 54) and LP (Eq. 56) are not changed. Only the constraints and models are changed. Disturbances are still unmeasured though their transfer functions to controlled variables have been changed. The modifications made on the Shell problem are summarized as follows:

Plant Model Error. The gains of the plant dynamic models are only 60% of the real plant. Other coefficients in the model are kept unchanged. Since the static model in LP formulation is derived from the plant dynamic model, it has the same gain error.

Constraint Change. The constraints on the manipulated variables are kept unchanged. However, in order to endure

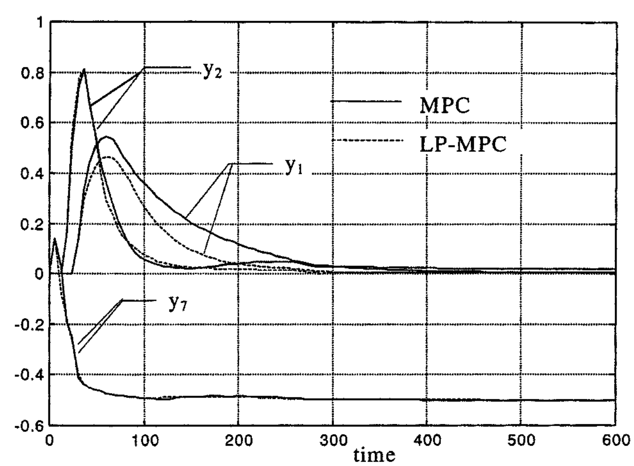

Figure 6. Responses of MPC and LP-MPC to disturbance change from $l_1 = l_2 = 0$ to $l_1 = l_2 = 0.4$.

large disturbances, the constraints on the input move and the controlled variables are extended to

$$
\begin{aligned}
&-1/6 \le \Delta u \le 1/6 \ \text{per min} \\
&-1 \le y_1 \le 1 \\
&-1 \le y_7 .
\end{aligned}
\tag{58}
$$

Disturbance Change. The new time constants in the transfer functions between l_1, l_2 and y_1, y_2, y_7 are the original time constants plus 100, that is, if the original transfer function is $[K/(1+\tau s)]e^{-\theta s}$, the new transfer function used in the simulation is $\{K/[1+(100+\tau)s]\}e^{-\theta s}$. The limit on the amplitude of the disturbance is ± 2. Both MPC and LP formulations contain modeling error.

Since there is modeling error and the prediction horizon is not infinity, the nominal stability theorems derived above do not apply here.

At time $t = 600$, the real-time optimizer knows through the global plant static relations that there is a step disturbance with final values of $l_1 = l_2 = 1.0$. (The real-time optimizer here used is the same as the LP formulation except that it works in an infrequent manner and perfect static models are used.) With initial disturbances $l_1 = l_2 = 0$ changing to $l_1 = l_2 = 1.0$, set points for y_1, y_2, and u_3 are changed from [0.005, 0.005, -0.32] to [0.005, 0.005, -0.44] by the real-time optimizer. After this set point transfer completes, the system will be at its steady state if it is stable. Then, at $t = 1{,}368$, the real-time optimizer is executed again with $l_1 = l_2 = -1.8$. It resets the set points on y_1, y_2, and u_3 from [0.005, 0.005, -0.44] to [-0.005, 0.005, 0.45]. This is a large change for u_3 and may cause the single-stage MPC control system to become unstable. For LP (QP)-MPC, the set point change is gradual rather than the sudden step change on the single-stage MPC. The response will be better.

Figure 7 shows the results of the single-stage MPC control system with ($w_y = 2$, $w_u = 1$). As can be seen for the first set point transfer, the system is stable because the set point change is small (from [0.005, 0.005, -0.32] to [0.005, 0.005, -0.44]). However, when the second set point change comes, the single-stage MPC control system is unable to maintain the stability of the system. The outputs and inputs oscillate significantly and the set point transfer is not successful. This is clearer if one looks at the manipulated variables.

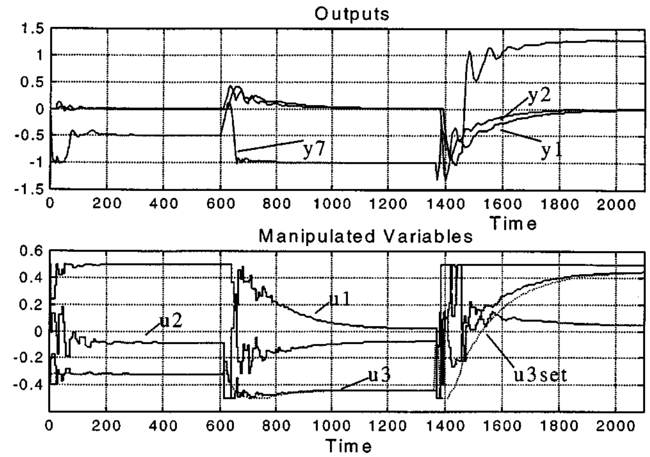

Figure 8. Global set point transfer of LP-MPC control system.

Figure 8 is the set point transfer dynamics for LP-MPC control system. It can be seen from the figure that both set point transfers are successful. For small set point changes, despite superior dynamic performance, LP-MPC does not differ much from single-stage MPC. For large set point changes, however, LP-MPC is stable and the dynamics is acceptable, while single-stage MPC tends to become unstable. It should be mentioned here that all the lower-level MPC parameters are the same as with single-stage MPC.

Both set point changes on u_3 for LP-MPC and single-stage MPC are plotted in Figure 9. It's clear that for LP-MPC, the change on set point is more gradual than that of the single-stage MPC. Also from the figure, it can be seen that there is no steady-state error on the set point of u_3 by using LP-MPC, even though there is modeling error in the LP formulation.

The process is maintained at a more economic state under LP-MPC. In Figure 9, consider $t = 1{,}368$. Because of the

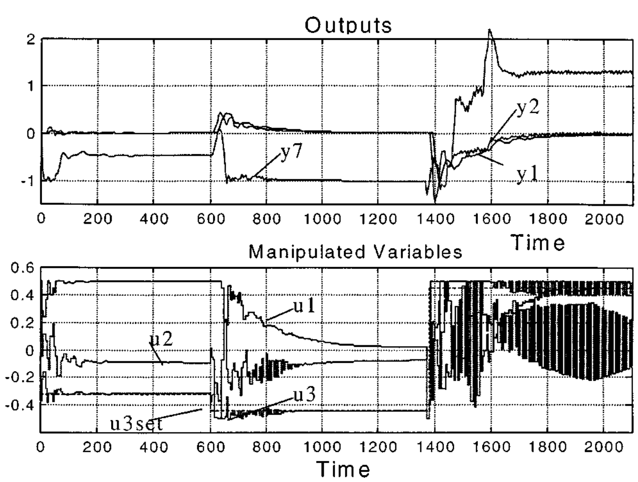

Figure 7. Global set point transfer of single-stage MPC control system.

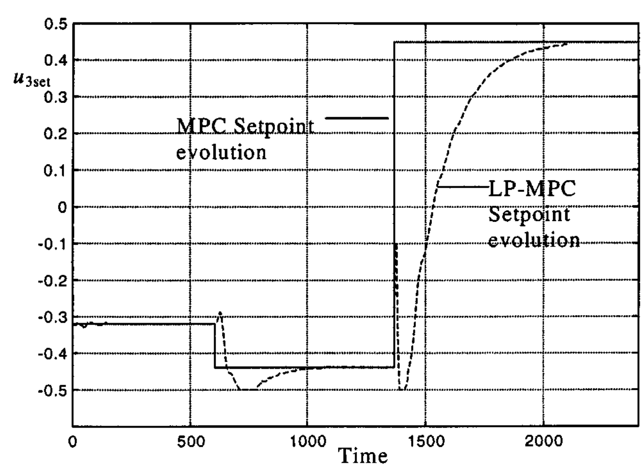

Figure 9. Evolution of set point of u_3 for LP-MPC and single-stage MPC.

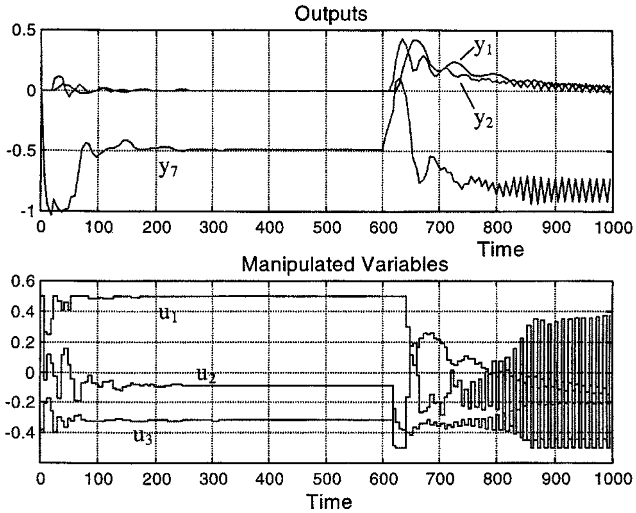

Figure 10. Response of single-stage MPC when disturbances change.

knowledge of the static disturbances, the real-time optimizer resets the set point of u_3 to 0.45. The set point from the RTO is larger than that from the upper-stage LP, resulting in profit loss. Although at $t = 600$, the real-time optimizer immediately gives out -0.44 as the set point of u_3, this set point is unreachable and is actually in the infeasible region. This can also be seen from Figure 7 that the actual response of u_3 does not go down immediately after $t = 600$. It maintains its original level until disturbances come in. Thus, there is no economic advantage of making the set point change in advance of disturbance in this case.

(vi) Performance when disturbances change in-between the RTO interval

The same modeling errors discussed above were used in this test. If disturbances change in between the RTO interval, there could be situations where the set points used by the single-stage MPC are inconsistent and/or infeasible. Figure 10 shows the response of single-stage MPC during the RTO interval (set points unchanged, $u_3^* = -0.32$) when disturbances change from $l_1 = l_2 = 0$ to $l_1 = l_2 = 1.0$ at time around 600. The single-stage MPC becomes unstable. The LP-MPC, however, is still stable (see Figure 11).

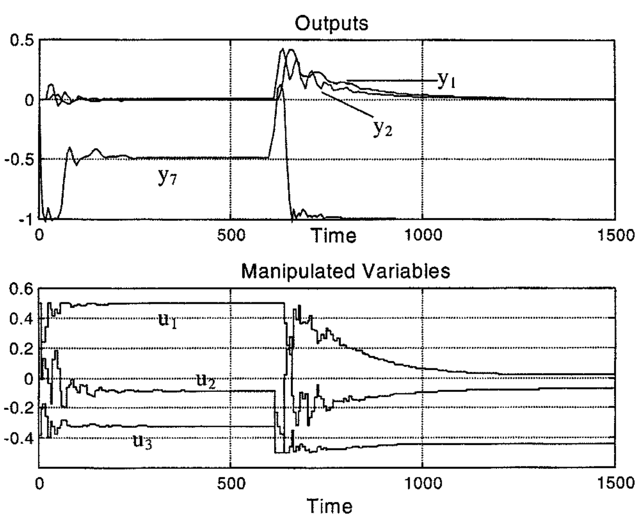

Figure 11. Response of LP-MPC when disturbances change.

Conclusion

In this article the properties of the two-stage MPC approach was examined, and this approach continuously estimates the effect of disturbances and revises set points. Nominal stability of the two-stage MPC was analyzed. Two-stage MPC directly addresses the issue of shifting economic optimum in presence of disturbances. It allows offset-free performance irrespective of the weights in the objective function. There are also some advantages in terms of robust stability and dynamic performance. These were demonstrated using the Shell control problem as a case study.

Acknowledgments

Support provided by NSF grant CTS-95-29578 is gratefully acknowledged. The authors would like to thank Srikanth Voorakaranam for his discussion and some early contributions on the Shell problem. Also, comments and suggestions from anonymous reviewers are greatly appreciated.

Notation

A, B, C = matrices in the state space model
f = plant wide economic objective function
g = inequality constraints
h = equality constraints
L = Lagrangian function
m = control horizon
p = prediction horizon
Q_y, R, S, P = penalty matrices in MPC formulation
$u, \Delta u, u_{set}$ = manipulated variables, change of manipulated variables and their set points
u_{max}, u_{min} = upper and lower limits of the manipulated variables
w_y, w_u = weights in the Shell problem MPC formulation
x = state variable
y, y_{set} = controlled variables and their set points
y_{max}, y_{min} = maximum and minimum output allowed
y_{ss}, u_{ss} = steady-state values for y and u
z, z_d = introduced in Eq. 26, change of z
ϵ = violation of hard constraints

Literature Cited

Brosilow, C., "Modular Multivariable Control Applied to the Shell Heavy Oil Fractionator Problem," *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 367 (1988).
Brosilow, C., and G. Q. Zhao, "A Linear Programming Approach to Constrained Multivariable Process Control," *Control Dyn. Syst.*, **27**, 141 (1988).
Cuthrell, J. E., D. E. Rivera, W. J. Schmidt, and J. A. Vegeais, "Solution to the Shell Standard Control Problem," *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 27 (1988).
Cutler, C. R., and B. L. Ramaker, "Dynamic Matrix Control—A Computer Control Algorithm," AIChE Meeting (Apr., 1979).
Forbes, J., and T. Marlin, "Model Accuracy for Economic Optimizing Controllers: The Bias Update Case," *Ind. Eng. Chem. Res.*, **33**, 1919 (1994).
Forbes, J., and T. Marlin, "Design Cost: A Systematic Approach to

Technology Selection for Model-Based Real-Time Optimization Systems,'' *Computers Chem. Eng.*, **20**, 717 (1996).
Garcia, C. E., and M. Morari, ''Internal Model Control: 2. Design Procedure for Multivariable Systems,'' *Ind. Eng. Chem. Process Des. Dev.*, **24**, 472 (1985a).
Garcia, C. E., and M. Morari, ''Internal Model Control: 3. Multivariable Control Law Computation and Tuning Guidelines,'' *Ind. Eng. Chem. Process Des. Dev.*, **24**, 484 (1985b).
Garcia, C. E., and A. M. Morshedi, ''Quadratic Programming Solution of Dynamic Matrix Control (QDMC),'' *Chem. Eng. Commun.*, **46**, 73 (1986).
Garcia, C. E., and D. M. Prett, ''Advances in Industrial Model—Predictive Control,'' *Chemical Process Control—CPC-III*, M. Morari and T. J. McAvoy, eds., CACHE and Elsevier, Amsterdam, p. 249 (1986).
Han, S. P., ''Superlinearly Convergent Variable Metric Algorithms for General Nonlinear Programming Problems,'' *Math. Programming*, **11**, 263 (1976).
Harkins, B. L., ''The DMC Controller,'' *ISA*, Paper No. 91-0427, 1991 Proceedings, Instrument Society of America, Research Triangle Park, NC, p. 853 (1991).
Holt, B. R., and Z. Lu, ''Scheduled Controller for Robust, Non-Linear Control: Introduction and Application to the Shell Standard Control Problem,'' *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 181 (1988).
Kailath, Thomas, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ (1980).
Kwakernaak, H., and R. Sivan, *Linear Optimal Control Systems*, Wiley Interscience, New York (1972).
Lee, J. H., and B. Cooley, ''Recent Advances in Model Predictive Control and Other Related Areas,'' CPC-V, Tahoe City, CA, CACHE and AIChE, p. 201 (1997).
Maciejowski, J. M., ''Robustness of Multivariable Smith Predictors,'' *J. of Process Control*, **4**, 29 (1994).
Manousiouthakis, V., ''A Game Theoretic Approach to Robust Controller Synthesis and the Shell Control Problem,'' *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 291 (1988).
Marlin, T. E., and A. N. Hrymak, ''Real-Time Operations Optimization of Continuous Processes,'' *Proc. of CPC V*, Tahoe City, CA (1996).
Martin, G. D., J. M. Caldwell, and T. E. Ayral, ''Predictive Control Applications for the Petroleum Refining Industry,'' *Japan Petrol. Inst. Petrol. Refining Conf.*, Tokyo, Japan (Oct., 1986).
McDonald, K., and A. Palazoglu, ''A Multiobjective Predictive Control Solution to the Shell Control Problem,'' *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 241 (1988).
Miletic, I., and T. Marlin, ''Results Analysis for Real-Time Optimization (RTO): Deciding When to Change the Plant Operation,'' *Computers Chem. Eng.*, **20**, 1077 (1996).
Moro, L. F. L., and D. Odloak, ''Constrained Multivariable Control of Fluid Catalytic Cracking Converters,'' *J. of Process Control*, **5**, 29 (1995).
Morshedi, A. M., C. R. Cutler, and T. A. Skrovanek, ''Optimal Solution of Dynamic Matrix Control with Linear Programming Techniques (LDMC),'' ACC '85, Boston, MA, p. 208 (1985).
Muske, K. R., ''Steady-State Target Optimization in Linear Model Predictive Control,'' *Amer. Control Conf.*, Albuquerque, NM, p. 3597 (1997).
Muske, K. R., and J. B. Rawlings, ''Model Predictive Control with Linear Models,'' *AIChE J.*, **39** (1993).
Palavajjhala, S., ''Studies in Model Predictive Control with Applications of Wavelet Transform,'' DSc. Thesis, Washington Univ., St. Louis, MO (Dec., 1994).
Powell, M. J. D., *Nonlinear Programming 3*, O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, eds., Academic Press, New York (1978).
Prett, D. M., and R. D. Gillette, ''Optimization and Constrained Multivariable Control of a Catalytic Cracking Unit,'' AIChE Meeting, Houston, TX (1979).
Prett, D. M., and C. E. Garcia, *Fundamental Process Control*, Butterworths, Stoneham, MA (1988).
Prett, D. M., C. E. Garcia, and B. L. Ramaker, *The Second Shell Process Control Workshop*, Butterworths, Stoneham, MA (1990).
Prett, D. M., and M. Morari, *The Shell Process Control Workshop*, Butterworths, Stoneham, MA, p. 355 (1986).
Rawlings, J. B., and J. W. Eaton, ''Optimal Control and Model Identification Applied to the Shell Standard Control Problem,'' *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 209 (1988).
Rawlings, J. B., and K. R. Muske, ''Stability of Constrained Receding Horizon Control,'' *IEEE Trans. Auto. Contr.*, **38**, 1512 (1993).
Richalet, J., A. Rault, J. L. Testud, and J. Papon, ''Model Predictive Heuristic Control: Applications to Industrial Processes,'' *Automatica*, **14**, 413 (1978).
Vuthandam, P., H. Genceli, and M. Nikolaou, ''Performance Bounds for Robust Quadratic Dynamic Matrix Control with End Condition,'' *AIChE J.*, **41**, 2083 (1995).
Wang, Tse-Wei, Chun-Yao Lien, J. D. Birdwell, and P. Hansen, ''Designing a Control System for the Shell Problem,'' *The Second Shell Process Control Workshop*, Butterworth, Stoneham, MA, p. 109 (1988).
Yousfi, C., and R. Tournier, ''Steady-State Optimization Inside Model Predictive Control,'' ACC '91, Boston, p. 1866 (1991).
Yu, Zheng H., Wei Li, Jay H. Lee, and M. Morari, ''State Estimation Based Model Predictive Control Applied to Shell Control Problem: A Case Study,'' *Chem. Eng. Sci.*, **49**, 285 (1994).
Zafiriou, E., ''Robust Model Predictive Control of Processes with Hard Constraints,'' *Computers & Chem. Eng. Proc. of AIChE Meeting and 1989 Amer. Control Conf.*, **14**, 359, Washington, DC (1990).
Zheng, A., and M. Morari, ''Stability of Model Predictive Control with Mixed Constraints,'' *IEEE Trans. on Automatic Control*, **40**, 1818 (1995).

Appendix: Transfer Function Shell Control Problem

First-order plus dead time (FOPDT) model: $[K/(1+\tau s)]e^{-\theta s}$

	Top Draw (u_1)	Side Draw (u_2)	Bottom Reflux Duty (u_3)	Inter. Reflux Duty (l_1)	Upper Reflux Duty (l_2)
Top End Point (y_1)	$K=4.05$ $\tau=50$ $\theta=27$	$K=1.77$ $\tau=60$ $\theta=28$	$K=5.88$ $\tau=50$ $\theta=27$	$K=1.20$ $\tau=45$ $\theta=27$	$K=1.44$ $\tau=40$ $\theta=27$
Side End Point (y_2)	$K=5.39$ $\tau=50$ $\theta=18$	$K=5.72$ $\tau=60$ $\theta=14$	$K=6.90$ $\tau=40$ $\theta=15$	$K=1.52$ $\tau=25$ $\theta=15$	$K=1.83$ $\tau=20$ $\theta=15$
Top Temp. (y_3)	$K=3.66$ $\tau=9$ $\theta=2$	$K=1.65$ $\tau=30$ $\theta=20$	$K=5.53$ $\tau=40$ $\theta=2$	$K=1.16$ $\tau=11$ $\theta=0$	$K=1.27$ $\tau=6$ $\theta=0$
Upper Reflux Temp. (y_4)	$K=5.92$ $\tau=12$ $\theta=11$	$K=2.54$ $\tau=27$ $\theta=12$	$K=8.10$ $\tau=20$ $\theta=2$	$K=1.73$ $\tau=5$ $\theta=0$	$K=1.79$ $\tau=19$ $\theta=0$
Side Draw Temp. (y_5)	$K=4.13$ $\tau=8$ $\theta=5$	$K=2.38$ $\tau=19$ $\theta=7$	$K=6.23$ $\tau=10$ $\theta=2$	$K=1.31$ $\tau=2$ $\theta=0$	$K=1.26$ $\tau=22$ $\theta=0$
Inter. Reflux Temp. (y_6)	$K=4.06$ $\tau=13$ $\theta=8$	$K=4.18$ $\tau=33$ $\theta=4$	$K=6.53$ $\tau=9$ $\theta=1$	$K=1.19$ $\tau=19$ $\theta=0$	$K=1.17$ $\tau=24$ $\theta=0$
Bottoms Reflux Temp. (y_7)	$K=4.38$ $\tau=33$ $\theta=20$	$K=4.42$ $\tau=44$ $\theta=22$	$K=7.20$ $\tau=19$ $\theta=0$	$K=1.14$ $\tau=27$ $\theta=0$	$K=1.26$ $\tau=32$ $\theta=0$

Manuscript received July 29, 1998, and revision received Apr. 9, 1999.